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Generalised gauge invariance of electromagnetism

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Abstract. Classical electromagnetism in the Lorentz gauge is reviewed from the standpoint of the variational principle. The gauge condition is imposed as a constraint on the Lagrangian density of the system using a Lagrange multiplier. A similar formulation is followed for the 'complete α-Lorentz gauge' of Yang. The uncoupled field equations in this gauge are derived and solved under simple boundary conditions. Without conforming to Maxwell's interpretation that electromagnetic radiation should propagate at speed c, we show that it must always do so regardless of the value of α. This is so because under the simple boundary conditions chosen, the electromagnetic potentials in the 'complete α-Lorentz gauge' are a gauge transformation of the first kind of the electromagnetic potentials in the Lorentz gauge. It is shown that electromagnetic radiation propagates at the invariant speed c under the most general of boundary conditions and under a more general type of gauge transformation. These classical results are generalised by brief reference to the Aharonov–Bohm effect. Finally, repercussions regarding advanced, as opposed to retarded, potentials and the Lorentz invariance of the formulation are considered.

1. Introduction

The 'complete α-Lorentz gauge' of Yang (1976) purported to combine the Lorentz gauge ($\alpha = 1$) and the Coulomb gauge ($\alpha = \infty$). In his paper, Yang derives field equations in the α-Lorentz gauge, and solves them by invoking Maxwell's criterion that electromagnetic radiation always propagates at speed c from charge and current densities. To do this he introduces the non-physical 'α-transverse current density' which acts as a source of radiation outside the source regions of the real, physical charge and current densities.

This paper looks at the formulation of electrodynamics in this 'complete α-Lorentz' gauge, deriving all equations from a variational principle. In §2 we show that Yang's field equations are in a mixed gauge corresponding to different choices of the Lagrange multiplier λ in the Lagrangian density of the system. The choice $\lambda = \alpha^2$ uncouples the field equations in the α-Lorentz gauge. These equations are taken to be the correct ones in this gauge, and do not involve unphysical charge or current densities. In §3 these equations are solved. In §4 the solutions are observed to be a gauge transformation of the Lorentz gauge. We show that the non-Lorentz parts of the α-Lorentz potentials are generated by the Lorentz parts, and that the electric and magnetic fields are independent of α, which remains arbitrary. In §5 we prove these statements for a general set of initial and boundary conditions, and in §6 we consider whether or not Lorentz covariance limits the arbitrary nature of α.

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Given the non-physicality of the electromagnetic potentials and the fact that Maxwell's equations are intrinsically Lorentz covariant this paper reintroduces the notion of superluminal transfer of information. Such propagation is not observable, and therefore should not be taken too literally, since it does not alter the nature of the observable electromagnetic field. The controversy over faster-than-light propagation, and the particles (tachyons) which carry such signals, has lasted for many years (Feldman 1974). Theories of tachyons (mostly scalar tachyons) are built around Lorentz covariance, tachyons being a class of particles with imaginary mass (Bilaniuk *et al* 1962). Imposition of Lorentz covariance has led to some remarkable properties for tachyons: non-localisability (Ecker 1970), the possibility of acausal effects (Newton 1967, Fox *et al* 1969), non-Lorentz invariance of the vacuum (Feinberg 1969), non-conservation of tachyon number under a Lorentz boost (Feinberg 1978). Even supersymmetric tachyons, it appears, have their problems (Li and Lu 1987). Elucidation of the theoretical properties of tachyons seems as elusive as the particle itself, although recent experimental evidence may suggest that at least one of the known neutrinos might possibly be a fermionic tachyon (Chodos *et al* 1985). More recently still, there has been some discussion about the possibility of superluminal transfer of information by superoptic wavepackets (Band 1988).

Although this paper deals with classical electromagnetism our 'tachyon field' (the non-Lorentz parts of the α-Lorentz potentials) has associated with it similar properties to those above; non-localisability (§3), the inclusion of advanced ($\alpha < 0$) as well as retarded potentials (§5), the non-Lorentz invariance of the α-Lorentz gauge condition (§6) and the non-conservation of the current associated with the tachyon potentials (§4).

There has been much discussion about how to reconcile the statistical requirements of quantum theory and the usual idea of causality (Stapp 1975, Cramer 1986). One suggestion is that superluminal connections may resolve the issue (Stapp 1977). If the electromagnetic field is associated with a non-observable, non-interacting tachyon field this may support such an interpretation of quantum mechanics.

2. Electromagnetism from a variational principle

Maxwell's equations for arbitrary charge and current densities $\rho(\boldsymbol{x},t)$, $\boldsymbol{J}(\boldsymbol{x},t)$ are

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{x},t) = 4\pi\rho(\boldsymbol{x},t) \tag{1a}$$

$$\nabla\times\boldsymbol{B}(\boldsymbol{x},t) = \frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{x},t) + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{x},t) \tag{1b}$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{x},t) \tag{2a}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{x},t) = 0 \tag{2b}$$

($\nabla \equiv \nabla_x$ unless otherwise stated), where the dielectric constant and the magnetic permeability have their vacuum values.

In order to know how the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ propagate we can deduce from (1) and (2) the wave equations given by

$$\Box_c^2\,\boldsymbol{E}(\boldsymbol{x},t) = 4\pi\left(\nabla\rho(\boldsymbol{x},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{J}(\boldsymbol{x},t)\right) \tag{3a}$$

$$\Box_c^2\,\boldsymbol{B}(\boldsymbol{x},t) = -\frac{4\pi}{c}\nabla\times\boldsymbol{J}(\boldsymbol{x},t) \tag{3b}$$

where

$$\Box_c^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{4}$$

For localised ρ and $\boldsymbol{J}$ without boundary surfaces, (3) have solutions

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= -\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t') \left(\nabla'\rho(\boldsymbol{x}', t') + \frac{1}{c^2}\frac{\partial}{\partial t'}\boldsymbol{J}(\boldsymbol{x}', t')\right) \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t')\ \nabla' \times \boldsymbol{J}(\boldsymbol{x}', t') \end{aligned} \tag{5}$$

$(\nabla' \equiv \nabla_{\boldsymbol{x}'})$, where

$$G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t') = \frac{\delta(t' + (\mid \boldsymbol{x} - \boldsymbol{x}' \mid)/c - t)}{\mid \boldsymbol{x} - \boldsymbol{x}' \mid} \tag{6}$$

is the retarded Green function of (4) and δ is the Dirac delta function.

Equally well one can introduce the concept of electromagnetic potentials by exploiting the homogeneity of (2):

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t) - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x}, t) \tag{7a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{A}(\boldsymbol{x}, t). \tag{7b}$$

Then the field equations for $\boldsymbol{A}$ and ϕ are, from (1),

$$\Box_c^2\,\phi(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial}{\partial t}\phi(\boldsymbol{x}, t)\right) = -4\pi\rho(\boldsymbol{x}, t) \tag{8a}$$

$$\Box_c^2\,\boldsymbol{A}(\boldsymbol{x}, t) - \nabla\left(\nabla\cdot\boldsymbol{A}(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial}{\partial t}\phi(\boldsymbol{x}, t)\right) = -\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{x}, t) \tag{8b}$$

which are consistent with the continuity equation

$$\nabla\cdot\boldsymbol{J}(\boldsymbol{x}, t) + \frac{\partial}{\partial t}\rho(\boldsymbol{x}, t) = 0. \tag{9}$$

Following the usual arguments (e.g. Jackson 1967, p179ff), $\boldsymbol{A}$ and ϕ are not unique and for a given set of potentials a further condition must be specified. This is the gauge condition of the form

$$g[\boldsymbol{A}, \phi] = 0. \tag{10}$$

The Lorentz gauge is of the form

$$g_o[\boldsymbol{A}_o, \phi_o] = \left[\nabla\cdot\boldsymbol{A}_o(\boldsymbol{x}, t) + \frac{1}{c}\frac{\partial}{\partial t}\phi_o(\boldsymbol{x}, t)\right] = 0 \tag{11}$$

where the subscript 'o' signifies the Lorentz gauge. In this gauge, (8) uncouple in $\boldsymbol{A}$ and ϕ giving

$$\square_c^2 \phi_o(\boldsymbol{x},t) = -4\pi\rho(\boldsymbol{x},t) \tag{12a}$$

$$\square_c^2 \boldsymbol{A}_o(\boldsymbol{x},t) = -\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{x},t) \tag{12b}$$

which are consistent with (9) and (11). Equations (8) (which are equivalent to (1)), and (12) may be derived from a variational principle (VP) of the form

$$\delta\int_{t_o}^{t_1}\mathrm{d}t\int_V \mathrm{d}^3\boldsymbol{x}\, L(\boldsymbol{x},t) = 0 \tag{13}$$

where $L(\boldsymbol{x},t)$ is a Lagrangian density and the integral is subject to the boundary conditions that all virtual paths are of zero variation at the time end points t_1 and t_o, and on the surface S enclosing V (Mandelstam and Yourgrau 1968).

The Lagrangian density is a function of $\boldsymbol{x}$ and t through the components and the first derivatives (in time and space) of the components of $\boldsymbol{A}$ and ϕ, which are treated as independent. A first-order Euler–Lagrange VP is therefore appropriate (Akhiezer 1962). Consider the Lagrangian density

$$L = \frac{1}{8\pi}\left(-\nabla\phi_o - \frac{1}{c}\frac{\partial\boldsymbol{A}_o}{\partial t}\right)^2 - \frac{1}{c}(\nabla\times\boldsymbol{A}_o)^2 + (\boldsymbol{J}\cdot\boldsymbol{A}_o/c - \rho\phi_o) - \frac{\lambda}{8\pi}\left(\nabla\cdot\boldsymbol{A}_o + \frac{1}{c}\frac{\partial\phi_o}{\partial t}\right)^2 \tag{14}$$

where all quantities are functions of $\boldsymbol{x}$ and t as above. The constraint has been included pre-multiplied by a dimensionless Lagrange multiplier $\lambda \equiv \lambda(\boldsymbol{x},t)$. The problem for the given constraint (the Lorentz gauge in this instance) is now completely specified by this Lagrangian (hence the subscript 'o'), to within this arbitrary multiplier λ.

Application of the VP gives

$$\begin{aligned}
\square_c^2\boldsymbol{A}_o + \nabla\left[(\lambda-1)\left(\nabla\cdot\boldsymbol{A}_o + \frac{1}{c}\frac{\partial\phi_o}{\partial t}\right)\right] &= -\frac{4\pi}{c}\boldsymbol{J}\\
\square_c^2\phi_o - \frac{1}{c}\frac{\partial}{\partial t}\left[(\lambda-1)\left(\nabla\cdot\boldsymbol{A}_o + \frac{1}{c}\frac{\partial\phi_o}{\partial t}\right)\right] &= -4\pi\rho.
\end{aligned} \tag{15}$$

Given the constraint (11), we see that $\lambda(\boldsymbol{x},t)$ does not affect the field equations for $\boldsymbol{A}_o$ and ϕ_o and therefore has no effect on the motion of the fields. We are therefore free to choose the value of λ which best suits our problem. If λ =0 we have (8) (or (1)) directly from the VP. This gauge, $\{g = g_o; \lambda = 0\}$, is called the Landau gauge (Itzykson and Zuber 1985). If λ=1 we have the uncoupled equations (12), direct from the VP. This gauge $\{g = g_o, \lambda = 1\}$ is called the Feynman gauge (Itzykson and Zuber 1985).

Suppose we now perform a gauge transformation of the second kind within the Lorentz gauge, on the Lagrangian density (14), i.e.

$$\begin{aligned}
\boldsymbol{A}_o(\boldsymbol{x},t) &\to \boldsymbol{A}_o(\boldsymbol{x},t) + \nabla\Lambda(\boldsymbol{x},t)\\
\phi_o(\boldsymbol{x},t) &\to \phi_o(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(\boldsymbol{x},t)
\end{aligned} \tag{16}$$

where $\Lambda(\boldsymbol{x}, t)$ is a continuous differentiable function of $\boldsymbol{x}$ and t. Then

$$L' = \frac{1}{8\pi}\left(-\nabla\phi_o - \frac{1}{c}\frac{\partial \boldsymbol{A}_o}{\partial t}\right)^2 - \frac{1}{8\pi}(\nabla \times \boldsymbol{A}_o)^2 + (\boldsymbol{J}\cdot\boldsymbol{A}_o/c - \rho\phi_o)$$
$$- \frac{\lambda}{8\pi}\left(\nabla\cdot\boldsymbol{A}_o + \frac{1}{c}\frac{\partial\phi_o}{\partial t} + \Box_c^2\Lambda\right)^2 + \frac{1}{c}\left(\boldsymbol{J}\cdot\nabla\Lambda + \rho\frac{\partial\Lambda}{\partial t}\right). \tag{17}$$

We have now introduced extra degrees of freedom into the Lagrangian density through the first and second time and space derivatives of Λ. Using a second-order Euler–Lagrange VP (Akhiezer 1962) , and allowing these derivatives of Λ to vary independently of all other quantities, we find

$$-\frac{1}{4\pi}\Box_c^2(\lambda\Box_c^2\Lambda) = \nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0. \tag{18}$$

Thus if we choose

$$\Box_c^2\Lambda(\boldsymbol{x}, t) = 0 \tag{19}$$

variation of $\boldsymbol{A}_o$ and ϕ_o still gives (15) and λ remains arbitrary.

We now turn our investigation to the complete α-Lorentz gauge of Yang (1976), i.e.

$$g = g_\gamma = \left[\nabla\cdot\boldsymbol{A}_\gamma(\boldsymbol{x}, t) + \frac{1}{\alpha^2 c}\frac{\partial}{\partial t}\phi_\gamma(\boldsymbol{x}, t)\right] = 0 \tag{20}$$

where α is an arbitrary constant and $\gamma = \alpha^2 - 1$. Using this gauge Yang derives the following field equations for $\boldsymbol{A}_\gamma$ and ϕ_γ:

$$\Box_c^2\boldsymbol{A}_\gamma(\boldsymbol{x}, t) = -\frac{4\pi}{c}\boldsymbol{J}_{\alpha t}(\boldsymbol{x}, t) \tag{21}$$
$$\Box_{\alpha c}^2\phi_\gamma(\boldsymbol{x}, t) = -4\pi\rho(\boldsymbol{x}, t)$$

where

$$\boldsymbol{J}_{\alpha t}(\boldsymbol{x}, t) = \boldsymbol{J}(\boldsymbol{x}, t) - \frac{\gamma}{4\pi\alpha^2}\frac{\partial}{\partial t}\nabla\phi_\gamma(\boldsymbol{x}, t) \tag{22}$$

is the α-transverse current density. He solves these equations using the Green functions of the self-adjoint scalar operators $\Box_c^2$, equation (4), and

$$\Box_{\alpha c}^2 = \nabla^2 - \frac{1}{\alpha^2 c^2}\frac{\partial^2}{\partial t^2} \tag{23}$$

i.e. $G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t')$, equation (6), and

$$G_{\alpha o}(\boldsymbol{x}, t \mid \alpha c \mid \boldsymbol{x}', t') = \frac{\delta(t' + (\mid \boldsymbol{x} - \boldsymbol{x}' \mid /\alpha c) - t)}{\mid \boldsymbol{x} - \boldsymbol{x}' \mid} \tag{24}$$

respectively.

Suppose we now consider the Lagrangian density

$$L = \frac{1}{8\pi}\left(-\nabla\phi_\gamma - \frac{1}{c}\frac{\partial \boldsymbol{A}_\gamma}{\partial t}\right)^2 - \frac{1}{8\pi}(\nabla\times\boldsymbol{A}_\gamma)^2 + (\boldsymbol{J}\cdot\boldsymbol{A}_\gamma/c - \rho\phi_\gamma) - \frac{\lambda}{8\pi}(\nabla\cdot\boldsymbol{A}_\gamma + \frac{1}{\alpha^2 c}\frac{\partial\phi_\gamma}{\partial t})^2. \tag{25}$$

We note that this Lagrangian is arbitrary with respect to $\lambda = \lambda(\boldsymbol{x}, t)$ and the constant α, g_γ being a 'family' of gauges, depending on the choice of α. Application of the VP gives

$$\Box_c^2 \boldsymbol{A}_\gamma + \nabla\left((\lambda-1)\nabla\cdot\boldsymbol{A}_\gamma + (\lambda-\alpha^2)\frac{1}{\alpha^2 c}\frac{\partial\phi_\gamma}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J} \tag{26}$$

$$\Box_c^2 \phi_\gamma - \frac{1}{\alpha^2 c}\frac{\partial}{\partial t}\left((\lambda-\alpha^2)\nabla\cdot\boldsymbol{A}_\gamma + (\lambda-\alpha^4)\frac{1}{\alpha^2 c}\frac{\partial\phi_\gamma}{\partial t}\right) = -4\pi\rho.$$

These equations are again consistent with (9) and (20). Noting the equation of the constraint (20), we see again that $\lambda(\boldsymbol{x}, t)$ remains completely arbitrary, and does not affect the motion of the fields $\boldsymbol{A}_\gamma$ and ϕ_γ. The same cannot be said of α.

It is also clear that (21) cannot be derived from a Lagrangian of the type (25). These equations are in a mixed gauge $\{g = g_\gamma, \lambda = 1\}$ for the first and $\{g = g_\gamma, \lambda = \alpha^2\}$ for the second. The equivalent gauge to the Feynman gauge $\{g = g_o, \lambda = 1\}$, i.e. the one which uncouples the field equations for $\boldsymbol{A}$ and ϕ, is in this case $\{g = g_\gamma, \lambda = \alpha^2\}$ giving

$$\Box_c^2 \boldsymbol{A}_\gamma(\boldsymbol{x}, t) + \gamma\nabla(\nabla\cdot\boldsymbol{A}_\gamma(\boldsymbol{x}, t)) = -\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{x}, t) \tag{27a}$$

$$\Box_{\alpha c}^2 \phi_\gamma(\boldsymbol{x}, t) = -4\pi\rho(\boldsymbol{x}, t) \tag{27b}$$

these equations being derived directly from the Lagrangian (25) with $\lambda = \alpha^2$. These we take to be the field equations of the electromagnetic potentials in the α-Lorentz gauge.

If we now perform a gauge transformation of the second kind on (25), and apply a second order Euler–Lagrange VP to the result, as above, we find that (26) remain unaltered and λ remains arbitrary if we choose

$$\Box_{\alpha c}^2 \Lambda(\boldsymbol{x}, t) = 0 \tag{28}$$

instead of (19).

We prefer to work with (27) because their right-hand sides are the real, physical charge and current densities, as opposed to the unphysical α-transverse current density. We also prefer to work with equations that can be derived directly from a VP. In the case of the classical motion of the fields, with Yang's restricted boundary conditions on $\boldsymbol{J}$ and ρ, this distinction between field equations is immaterial. However, if we were to attempt to quantise the fields, or relax the boundary conditions on $\boldsymbol{J}$ and ρ we must have a Lagrangian which gives the desired field equations directly.

3. Solution of field equations

For this section and the next we restrict the nature of the charge and current densities to maintain simplicity. Suppose $\boldsymbol{J}$ and ρ are localised without boundary surfaces and that both vanish identically before and at $t = t_o$.

The solution of (27*b*) for ϕ_γ remains that of Yang, namely

$$\phi_\gamma(\boldsymbol{x},t) = \int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3x' \; G_{\alpha o}(\boldsymbol{x},t \mid \alpha c \mid \boldsymbol{x}',t')\; \rho(\boldsymbol{x}',t') \tag{29}$$

(for $G_{\alpha o}$ see (24)). In order to solve (27*a*) we require a dyadic Green function $\Theta_\gamma(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')$ such that

$$\Box_c^2\, \Theta_\gamma(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') + \gamma\nabla(\nabla\cdot\Theta_\gamma(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')) = -4\pi\mathscr{F}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{30}$$

where $\mathscr{F}$ is the dyadic idemfactor. In order to solve (30) we have chosen a fourth-order method. Uncoupling (30) for the components of $\Theta_\gamma(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')$ we have

$$\Box_c^2\,\Box_{\alpha c}^2\,\Theta_\gamma(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = -4\pi\left(\mathscr{F}\Box_{\alpha c}^2\,\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') - \frac{\gamma}{\alpha^2 c}\nabla\nabla\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\right). \tag{31}$$

We now require the scalar Green function $G(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')$ of the self-adjoint scalar operator

$$\Box_c^2\,\Box_{\alpha c}^2 = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{1}{\alpha^2 c^2}\frac{\partial^2}{\partial t^2}\right) \tag{32}$$

(the space between the parallel lines in G signifies that there is no fixed velocity of propagation), i.e. we require

$$\Box_c^2\,\Box_{\alpha c}^2\,G(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \tag{33}$$

Given that

$$\Box_c^2\,G_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}',t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{34a}$$

$$\Box_{\alpha c}^2\,G_{\alpha o}(\boldsymbol{x},t \mid \alpha c \mid \boldsymbol{x}',t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{34b}$$

we impose the following boundary conditions on G:

$$\Box_{\alpha c}^2\,G(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = G_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}',t') \tag{35}$$

$$\Box_c^2\,G(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = G_{\alpha o}(\boldsymbol{x},t \mid \alpha c \mid \boldsymbol{x}',t'). \tag{36}$$

These boundary conditions ensure that G obeys the causality condition and is well behaved at large $|\,\boldsymbol{x}-\boldsymbol{x}'\,|$, i.e.

$$G = 0 \quad \text{for} \quad t < t' \qquad G \to 0 \quad \text{as} \quad |\,\boldsymbol{x}-\boldsymbol{x}'\,| \to \infty$$

respectively. In order to find an explicit expression for G we note that the solution of (36) in infinite space with no boundary surfaces is

$$G(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = -\frac{1}{c}\int \mathrm{d}t'' \int \mathrm{d}^3x''\; G_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}'',t'')\; G_{\alpha o}(\boldsymbol{x}'',t'' \mid \alpha c \mid \boldsymbol{x}',t'). \tag{37}$$

Substituting (37) into (33) we easily show that this integral expression is a solution. The integration over $t' \in [t_o,\infty]$ is straightforward. The integral over the volume element

d^3x'' is separable in prolate spheroidal coordinates $\phi \in [0, 2\pi]$, $\mu \in [-1, 1]$, $\lambda \in [1, \infty]$. The result is

$$G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') = \frac{\alpha^2 c^2}{\gamma R}[M(\tau, R/\alpha c) - M(\tau, R/c)] \tag{38}$$

where $\tau = t - t'$ and $R = | \boldsymbol{x} - \boldsymbol{x}' |$ and we have used the integral representations (A1.7) and (A1.10) of the δ function and the minimum function respectively. It is straightforward to show that (38) satisfies the boundary conditions (35) and (36) and also (33)†. Uncoupling (27a) for $\boldsymbol{A}_\gamma$ we have

$$\Box_c^2 \Box_{\alpha c}^2 \boldsymbol{A}_\gamma(\boldsymbol{x}, t) = -\frac{4\pi}{c}\bar{\boldsymbol{J}}(\boldsymbol{x}, t) \tag{39}$$

where

$$\bar{\boldsymbol{J}}(\boldsymbol{x}, t) = \Box_{\alpha c}^2 \boldsymbol{J}(\boldsymbol{x}, t) - \frac{\gamma}{\alpha^2}\nabla(\nabla \cdot \boldsymbol{J}(\boldsymbol{x}, t)). \tag{40}$$

Hence one possible solution for $\boldsymbol{A}_\gamma$ is

$$\boldsymbol{A}_\gamma(\boldsymbol{x}, t) = \frac{1}{c}\int_{t_o}^{t_1} dt' \int_V d^3x'\, G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t')\, \bar{\boldsymbol{J}}(\boldsymbol{x}', t'). \tag{41}$$

We prefer the solution in terms of $\boldsymbol{J}(\boldsymbol{x}, t)$ instead of $\bar{\boldsymbol{J}}(\boldsymbol{x}, t)$. We notice from (33) that

$$\Box_c^2 \Box_{\alpha c}^2 \mathscr{F} G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') = -4\pi \mathscr{F} \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'). \tag{42}$$

Comparing with equation (31) we then have

$$\Theta_\gamma(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') = \mathscr{F} \Box_{\alpha c}^2\, G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') - \frac{\gamma}{\alpha^2}\nabla\nabla G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') \tag{43}$$

$$= \Theta_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t') + \tilde{\Theta}_\alpha(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') \tag{44}$$

where

$$\Theta_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t') = \mathscr{F} \Box_{\alpha c}^2\, G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') \tag{45}$$

$$= \mathscr{F} G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t') \tag{46}$$

from (35), and

$$\tilde{\Theta}_\alpha(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') = -\frac{\gamma}{\alpha^2}\nabla\nabla G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') \tag{47}$$

$$= \left[\frac{\hat{\boldsymbol{R}}\hat{\boldsymbol{R}}\, \delta(\tau - R/\alpha' c)}{\alpha'^2} + (\mathscr{F} - 3\hat{\boldsymbol{R}}\hat{\boldsymbol{R}})\,\frac{c^2\tau}{R^3}\, H(R/\alpha' c - \tau)\right]_1^\alpha \tag{48}$$

† It may also be shown that $G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t')$ obeys the same reciprocity relation as $G_o(\boldsymbol{x}, t \mid c \mid \boldsymbol{x}', t')$, i.e.

$$G(\boldsymbol{x}, t \parallel \boldsymbol{x}', t') = G(\boldsymbol{x}', -t' \parallel \boldsymbol{x}, -t).$$

where H is the Heaviside step function (see (A1.2)), and

$$\hat{\boldsymbol{R}} = \frac{\boldsymbol{R}}{R} \qquad [f(\alpha')]_1^\alpha = f(\alpha) - f(1).$$

Hence the solution of (27*a*) is, from (30) and (44),

$$\boldsymbol{A}_\gamma(\boldsymbol{x}, t) = \frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \Theta_\gamma(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t') \cdot \boldsymbol{J}(\boldsymbol{x}', t'). \tag{49}$$

We note that we can resolve the projection of Θ_γ on $\boldsymbol{J}$, i.e. $\Theta_\gamma \cdot \hat{\boldsymbol{J}}$, into two components, one parallel to $\hat{\boldsymbol{R}}$, namely $\hat{\boldsymbol{R}}(\hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{J}})$, and the other perpendicular to $\hat{\boldsymbol{R}}$, namely $\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{J}} \times \hat{\boldsymbol{R}})$. They are respectively

$$\frac{\delta(\tau - R/c)}{R} + \frac{c^2\tau}{R^3}[H(R/\alpha'c - \tau)]_1^\alpha \tag{50}$$

$$\frac{\delta(\tau - R/\alpha'c)}{R} - \frac{2c^2\tau}{R^3}[H(R/\alpha'c - \tau)]_1^\alpha. \tag{51}$$

For large R or large τ, i.e. $t \gg t'$, the second term in each component tends to zero, and is zero, respectively.

The transverse modes of vibration are then associated with causal propagation at speed c, and the longitudinal ones with speed αc. This second term thus acts to couple the different speeds of propagation and an α-dependent term appears to be associated with the transverse modes of vibration. (This is in contrast to the free-field case, i.e. $\boldsymbol{J} = \boldsymbol{0}$, $\rho = 0$, when the transverse vibrations are propagated only at speed c, and the longitudinal ones only at αc.) It is therefore by no means obvious that the observable electric and magnetic fields are independent of α.

We also note that the part of the Green function containing the Heaviside step function introduces an element of non-localisability into the tachyon potentials. In the Lorentz gauge, only disturbances originating at time $t - R/c$ contribute to the vector potential at time t. In our case there is a cumulative effect where all disturbances between times $t - R/c < t' < t - R/\alpha c$ contribute to this potential at time t. The motion of the vector potential in space is like that of a 'smoke ring'†.

4. The α dependence of the E and B fields

Equation (44) suggests the decomposition for $\boldsymbol{A}_\gamma$ and ϕ_γ

$$\boldsymbol{A}_\gamma(\boldsymbol{x}, t) = \boldsymbol{A}_o(\boldsymbol{x}, t) + \tilde{\boldsymbol{A}}_\alpha(\boldsymbol{x}, t) \tag{52a}$$

$$\phi_\gamma(\boldsymbol{x}, t) = \phi_o(\boldsymbol{x}, t) + \tilde{\phi}_\alpha(\boldsymbol{x}, t). \tag{52b}$$

† The field equation for $\boldsymbol{A}_\gamma$ may be written in the form

$$\alpha^2c^2\nabla(\nabla \cdot \boldsymbol{A}_\gamma(\boldsymbol{x}, t)) - c^2\nabla \times (\nabla \times \boldsymbol{A}_\gamma(\boldsymbol{x}, t)) - \frac{\partial^2}{\partial t^2}\boldsymbol{A}_\gamma(\boldsymbol{x}, t) = -4\pi c\,\boldsymbol{J}(\boldsymbol{x}, t).$$

This is the time-dependent equation for elastic waves in an isotropic medium. For $t \gg t'$ the longitudinal vibrations of velocity αc, and the transverse vibrations of velocity c, uncouple. Under these conditions the motion near $\boldsymbol{x} = \boldsymbol{x}'$ is like that of an expanding 'smoke ring', the outer circumference propagating at velocity αc, the inner at velocity c. (Assuming $\alpha > 1$.)

We have used the subscript 'o' here to denote the Lorentz gauge (see §2) since A_o and ϕ_o obey the Lorentz condition (11), and the field equations (12). It is apparent from this last fact that it is the Lorentz part of A_γ and ϕ_γ which couples to the real external conserved charge and current density. We write solutions of (52a) in dyadic form thus:

$$A_o(\boldsymbol{x},t) = \frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \Theta_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}',t')\cdot \boldsymbol{J}(\boldsymbol{x}',t') \tag{53a}$$

$$\tilde{A}_\alpha(\boldsymbol{x},t) = \frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \tilde{\Theta}_\alpha(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')\cdot \boldsymbol{J}(\boldsymbol{x}',t') \tag{53b}$$

where Θ_o and $\tilde{\Theta}_\alpha$ are defined in (46) and (48). The solutions of (52b) are

$$\phi_o(\boldsymbol{x},t) = \int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ G_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}',t')\,\rho(\boldsymbol{x}',t') \tag{54a}$$

$$\tilde{\phi}_\alpha(\boldsymbol{x},t) = \int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \tilde{G}_\alpha(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')\,\rho(\boldsymbol{x}',t') \tag{54b}$$

where G_o is defined in (6) and

$$\tilde{G}_\alpha(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = G_{\alpha o}(\boldsymbol{x},t \mid \alpha c \mid \boldsymbol{x}',t') - G_o(\boldsymbol{x},t \mid c \mid \boldsymbol{x}',t') \tag{55}$$

$$= \frac{\delta(\tau - R/\alpha c)}{R} - \frac{\delta(\tau - R/c)}{R}. \tag{56}$$

If we now define the quantity $\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')$ as

$$\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = c\int_1^\alpha \frac{\mathrm{d}\alpha'}{\alpha'^2} H(\tau - R/\alpha' c) \tag{57}$$

then

$$\tilde{\Theta}_\alpha(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = \nabla\nabla\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') \tag{58a}$$

$$\tilde{G}_\alpha(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') \tag{58b}$$

and we may now write (52a) in the following form:

$$\begin{aligned} A_\gamma(\boldsymbol{x},t) &= A_o(\boldsymbol{x},t) + \nabla\frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \nabla'\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')\cdot \boldsymbol{J}(\boldsymbol{x}',t') \\ &= A_o(\boldsymbol{x},t) + \nabla\frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \nabla'\cdot \boldsymbol{J}(\boldsymbol{x}',t')\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') \\ &\quad - \nabla\frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \oint_S \mathrm{d}a'\ \boldsymbol{J}(\boldsymbol{x}',t')\cdot\hat{\boldsymbol{n}}\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') \end{aligned} \tag{59}$$

where $\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{n}}(\boldsymbol{x}')$ is the outward normal to the surface S enclosing V. Under present boundary conditions this surface integral is zero. Following a similar procedure for (52b) we find

$$\begin{aligned} \phi_\gamma(\boldsymbol{x},t) &= \phi_o(\boldsymbol{x},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\ \frac{\partial}{\partial t'}\rho(\boldsymbol{x}',t')\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t') \\ &\quad + \frac{1}{c^2}\frac{\partial}{\partial t}\int_V \mathrm{d}^3\boldsymbol{x}'(\rho(\boldsymbol{x}',t')\,\Omega(\boldsymbol{x},t \,\|\, \boldsymbol{x}',t')\,|_{t'=t_o} \end{aligned} \tag{60}$$

the final term again being zero under present boundary conditions. Thus from (59), (60) we obtain

$$A_\gamma(x,t) = A_o(x,t) + \nabla\chi(x,t) \tag{61}$$

$$\phi_\gamma(x,t) = \phi_o(x,t) - \frac{1}{c}\frac{\partial}{\partial t}\chi(x,t)$$

where

$$\chi(x,t) = -\frac{1}{c}\int_{t_o}^{t_1} dt' \int_V d^3x' \frac{\partial}{\partial t'}\rho(x',t')\,\Omega(x,t \,||\, x',t'). \tag{62}$$

It is now clear that A_γ, ϕ_γ and A_o, ϕ_o are related by a gauge transformation of the first kind under present boundary conditions, since

$$\square^2_{\alpha c}\,\chi(x,t) = -\left[\nabla\cdot A_o(x,t) + \frac{1}{\alpha^2 c}\frac{\partial}{\partial t}\phi_o(x,t)\right] \tag{63}$$

from (61). Given (61) $\boldsymbol{E}$ and $\boldsymbol{B}$ are trivially independent of α under present boundary conditions, i.e.

$$B_\gamma(x,t) = \nabla\times A_\gamma(x,t) \qquad\qquad = B_o(x,t)$$

$$E_\gamma(x,t) = -\nabla\phi_\gamma(x,t) - \frac{1}{c}\frac{\partial}{\partial t}A_\gamma(x,t) = E_o(x,t).$$

Thus the electric and magnetic fields are always propagated at speed c regardless of the value of α, i.e. all the macroscopic properties of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are unaffected by α. This further arbitrariness in the electromagnetic potentials stems from the fact that the Lagrangian density of the system, (25), is arbitrary with respect to α.

In the Aharonov–Bohm effect (Aharonov and Bohm 1959), the phase of the electronic wavefunction for any closed trajectory C is given by $(q/\hbar)\oint_C A_o\cdot ds$, where q is the charge, $\hbar$ is Planck's constant, ds is the vector elemental path length and A_o is the usual vector potential $(\phi/2\pi\rho)\hat{\theta}$. Here ϕ is the total magnetic flux, ρ is the radial coordinate and $\hat{\theta}$ the transverse unit vector. The singularity at $\rho = 0$, the line of magnetic flux, is responsible for Stokes' theorem failing to predict a null effect. However, application of Stokes' theorem to $\nabla\chi$ round C does predict a null effect since χ given by (62) is non-singular, irrespective of the value of α.

Consider again the decomposition (52). Substituting these into the α-Lorentz gauge condition (20) we have, given (11),

$$\nabla\cdot\tilde{A}_\alpha + \frac{1}{\alpha^2 c}\frac{\partial\tilde{\phi}_\alpha}{\partial t} = \frac{\gamma}{\alpha^2 c}\frac{\partial\phi_o}{\partial t}. \tag{64}$$

Using (52) again in (27), we have

$$\square^2_c\,\tilde{A}_\alpha + \gamma\nabla(\nabla\cdot\tilde{A}_\alpha) = -\frac{4\pi}{c}\tilde{J}_\alpha \tag{65a}$$

$$\square^2_{\alpha c}\,\tilde{\phi}_\alpha = -4\pi\tilde{\rho}_\alpha \tag{65b}$$

where

$$\tilde{J}_\alpha = \frac{c\gamma}{4\pi}\nabla(\nabla\cdot A_o) \qquad (66a)$$

$$\tilde{\rho}_\alpha = \frac{\gamma}{4\pi\alpha^2c^2}\frac{\partial^2\phi_o}{\partial t^2}. \qquad (66b)$$

Clearly

$$\nabla\cdot\tilde{J}_\alpha + \frac{\partial\tilde{\rho}_\alpha}{\partial t} = -\frac{\gamma}{4\pi}\Box^2_{\alpha c}\frac{\partial\phi_o}{\partial t} \neq 0. \qquad (67)$$

Equations (65) may be taken to be the field equations of the α-dependent parts of A_γ and ϕ_γ, i.e. $\tilde{A}_\alpha$ and $\tilde{\phi}_\alpha$, what we might call the 'tachyon potentials'. We note from (67) that these potentials are associated with a non-conserved current which depends on A_o and ϕ_o (rather than on J and ρ explicitly). Thus it appears that A_o and ϕ_o act as a source generating the tachyon potentials $\tilde{A}_\alpha$ and $\tilde{\phi}_\alpha$. We can see this more clearly from (41) by replacing $\nabla'(\nabla'\cdot J(x',t'))$ by $-(c/4\pi)\Box'^2_c[\nabla'(\nabla'\cdot A_o(x',t'))]$ from (12b), and integrating by parts to obtain, under present boundary conditions,

$$\begin{aligned} A_\gamma(x,t) &= \frac{1}{c}\int_{t_o}^{t_1}\mathrm{d}t'\int_V \mathrm{d}^3x'\left(G_o(x,t\mid c\mid x',t')\,J(x',t') + \frac{1}{\alpha^2}G_{\alpha o}(x,t\mid \alpha c\mid x',t')\,\tilde{J}_\alpha(x',t')\right) \\ &= \frac{1}{c}\int_V\frac{\mathrm{d}^3x'}{R}\left(J(x',t-R/c) + \frac{1}{\alpha^2}\tilde{J}_\alpha(x',t-R/\alpha c)\right). \end{aligned} \qquad (68)$$

Thus the external current density J generates the potential A_o (the first term in (68)), then A_o (acting through $\tilde{J}_\alpha$) generates the potential $\tilde{A}_\alpha$ (the second term). Note that the arguments of J and $\tilde{J}_\alpha$ are in agreement with this interpretation, the disturbance generating $\tilde{A}_\alpha$ happening after that generating A_o, but being propagated faster, both effects being 'felt' simultaneously by A_γ.

Suppose we now perform a gauge transformation of the second kind on (61), i.e.

$$\begin{aligned} A_\gamma(x,t) &\longrightarrow A_\gamma(x,t) + \nabla\Lambda(x,t) \\ \phi_\gamma(x,t) &\longrightarrow \phi_\gamma(x,t) - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(x,t). \end{aligned} \qquad (69)$$

Is it possible to force A_γ and ϕ_γ into the Lorentz gauge by an appropriate choice of Λ ? It appears from (61) that $\Lambda = -\chi$ is a possibility. However, from §2 we know that Λ must obey (28), but χ satisfies (63). Therefore there is no gauge transformation of the second kind which transforms between the Lorentz and α-Lorentz gauges. This generalises a conclusion reached by Yang.

5. The initial-value boundary-surface problem

It may appear from (59) and (60) that for non-vanishing initial values of J and ρ, and for non-vanishing surfaces, (61) no longer hold and E_γ becomes α dependent. We now relax the restrictions of §3 and solve the initial-value boundary-surface problem. Our approach for A_γ is through the fourth-order Green function (38), and the bilinear concomitant, of the self-adjoint scalar operator $\Box_c^2\,\Box_{\alpha c}^2$, equation (32). The initial-value boundary-surface solution for A_γ in dyadic form is written down in appendix 2.

The bilinear concomitant of the self-adjoint scalar operator $\Box_c^2\,\Box_{\alpha c}^2$ for the scalars u, v is

$$S = [u\nabla(\nabla^2 v) - v\nabla(\nabla^2 u) - (\nabla^2 v)\nabla u + (\nabla^2 u)\nabla v] + \left(1+\frac{1}{\alpha^2}\right)\frac{1}{c^2}\left(u\frac{\partial}{\partial t}\nabla v - v\frac{\partial}{\partial t}\nabla u\right) \tag{70}$$

$$T = -\left(1+\frac{1}{\alpha^2}\right)\frac{1}{c^2}\left(u\frac{\partial}{\partial t}(\nabla^2 v) - v\frac{\partial}{\partial t}(\nabla^2 u)\right) + \frac{1}{\alpha^2 c^4}\left(u\frac{\partial^3 v}{\partial t^3} - v\frac{\partial^3 u}{\partial t^3} - \frac{\partial u}{\partial t}\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}\frac{\partial^2 u}{\partial t^2}\right) \tag{71}$$

such that

$$u\Box_c^2\,\Box_{\alpha c}^2\, v - v\Box_c^2\,\Box_{\alpha c}^2\, u = \nabla\cdot S + \frac{\partial T}{\partial t} \tag{72}$$

or, in integral form,

$$\int_{t_o}^{t_1} dt' \int_V d^3x'\ (u\Box_c'^2\,\Box_{\alpha c}'^2\, v - v\Box_c'^2\,\Box_{\alpha c}'^2\, u) = -\int_V d^3x'\ (T\,|_{t'=t_o} + \int_{t_o}^{t_1} dt' \oint_S da'\ [\hat{n}\cdot S] \tag{73}$$

where $\hat{n} \equiv \hat{n}(x')$ is the outward normal to the boundary surface S enclosing V, and we have anticipated the fact that $T(t_1) = 0$. Consider

$$\int_{t_o}^{t_1} dt' \int_V d^3x'\ (G\Box_c'^2\,\Box_{\alpha c}'^2\, A_\gamma' - A_\gamma'\Box_c'^2\,\Box_{\alpha c}'^2\, G) \tag{74}$$

where $G \equiv G(x, t\,||\,x', t')$ is the fourth-order Green function (38) and where the primes denote the arguments and operators as functions of x' and t'. We note that in (70) and (71) S is now a dyadic, and T, u and v are vectors. From (33) and (39), equation (74) becomes, using the integral theorems of appendix 2 and (35) and (36),

$$\begin{aligned} 4\pi A_\gamma(x,t) &- \frac{4\pi}{c}\int_{t_o}^{t_1} dt' \int_V d^3x'\ \left(G_o J' - \frac{\gamma}{\alpha^2} G\nabla'(\nabla'\cdot J')\right) \\ &- \frac{4\pi}{c}\int_{t_o}^{t_1} dt' \oint_S da'[G(\hat{n}\cdot\nabla')J' - J'(\hat{n}\cdot\nabla' G)] \\ &- \frac{4\pi}{\alpha^2 c^3}\int_V d^3x' \left(G\frac{\partial J'}{\partial t'} - J'\frac{\partial G}{\partial t'}\right|_{t'=t_o}. \end{aligned} \tag{75}$$

This last term follows from the fact that $G = 0$ at $t' = t_1 > t$. Using the boundary conditions (35), (36) for G we can simplify the expressions for the dyadic S and the vector $\boldsymbol{T}$ in (70), (71). Using the generalised Green theorem (73) we then find

$$\begin{aligned}
\boldsymbol{A}_\gamma(\boldsymbol{x},t) = {} & \frac{1}{c}\int_{t_o}^{t_1}\mathrm{d}t'\int_V \mathrm{d}^3x' \left(G_o\boldsymbol{J}' - \frac{\gamma}{\alpha^2}G\nabla'(\nabla'\cdot\boldsymbol{J}')\right) \\
& + \frac{1}{4\pi c^2}\int_V \mathrm{d}^3x' \left[G_o\left(\frac{\partial \boldsymbol{A}'}{\partial t'}\right)_T - (\boldsymbol{A}')_T\frac{\partial G_o}{\partial t'}\right|_{t'=t_o} \\
& - \frac{\gamma}{4\pi\alpha^2c^2}\int_V \mathrm{d}^3x' \left[G\nabla'\left\{\nabla'\cdot\left[\left(\frac{\partial \boldsymbol{A}'}{\partial t'}\right)_T\right]\right\} - \nabla'[\nabla'\cdot(\boldsymbol{A}')_T]\frac{\partial G}{\partial t'}\right|_{t'=t_o} \\
& - \frac{1}{4\pi}\int_{t_o}^{t_1}\mathrm{d}t'\oint_S \mathrm{d}a'\,[G(\hat{\boldsymbol{n}}\times(\nabla'\times\boldsymbol{A}'))_S + (\hat{\boldsymbol{n}}\times\boldsymbol{A}')_S\times\nabla'G + (\hat{\boldsymbol{n}}\cdot\boldsymbol{A}')_S\nabla'G] \\
& + \frac{\gamma}{4\pi}\int_{t_o}^{t_1}\mathrm{d}t'\oint_S \mathrm{d}a'\,[\nabla'(\nabla'\cdot\boldsymbol{A}')_S(\hat{\boldsymbol{n}}\cdot\nabla'G) + G\hat{\boldsymbol{n}}\nabla'^2(\nabla'\cdot\boldsymbol{A}')_S].
\end{aligned} \tag{76}$$

The initial-value problem is specified by the initial values of $\boldsymbol{A}_\gamma$ and $\partial\boldsymbol{A}_\gamma/\partial t$ at time t_o. These are denoted by $(\boldsymbol{A}')_T$ and $(\partial\boldsymbol{A}'/\partial t')_T$ respectively†. The boundary-surface problem may be specified by different sets of boundary conditions. Two possibilities $(\hat{\boldsymbol{n}}\cdot\boldsymbol{A}')_S$ and $(\hat{\boldsymbol{n}}\times\boldsymbol{A}')_S$ correspond to fixing the normal and tangential components of the field $\boldsymbol{A}_\gamma$ at the boundary surface. Others, $(\hat{\boldsymbol{n}}\times(\nabla'\times\boldsymbol{A}'))_S$ and $(\nabla'\cdot\boldsymbol{A}')_S$, correspond to fixing gradients of the field $\boldsymbol{A}_\gamma$ at the boundary surface.

The general solution of the field equation for ϕ_γ is much simpler:

$$\begin{aligned}
\phi_\gamma(\boldsymbol{x},t) = {} & \int_{t_o}^{t_1}\mathrm{d}t'\int_V \mathrm{d}^3x'\,G_{\alpha o}\rho' + \frac{1}{4\pi\alpha^2c^2}\int_V \mathrm{d}^3x' \left[G_{\alpha o}\left(\frac{\partial\phi'}{\partial t'}\right)_T - (\phi')_T\frac{\partial G_{\alpha o}}{\partial t'}\right|_{t'=t_o} \\
& + \frac{1}{4\pi}\int_{t_o}^{t_1}\mathrm{d}t'\oint_S \mathrm{d}a'\,[G_{\alpha o}(\hat{\boldsymbol{n}}\cdot\nabla'\phi')_S - (\phi')_S(\hat{\boldsymbol{n}}\cdot\nabla'G_{\alpha o})]
\end{aligned} \tag{77}$$

using a similar notation to (76) for initial and boundary conditions.

We are now in a position to calculate $\boldsymbol{B}_\gamma$ and $\boldsymbol{E}_\gamma$ by allowing the initial conditions to vary in $\boldsymbol{x}'$ and t', and by using (7). Taking the curl of (76) we find that

$$\begin{aligned}
\boldsymbol{B}_\gamma(\boldsymbol{x},t) = {} & \frac{1}{c}\int_{t_o}^{t_1}\mathrm{d}t'\int_V \mathrm{d}^3x'\,G_o(\nabla'\times\boldsymbol{J}') \\
& + \frac{1}{4\pi c^2}\int_V \mathrm{d}^3x' \left(G_o\left(\frac{\partial\boldsymbol{B}'}{\partial t'}\right)_T - (\boldsymbol{B}')_T\frac{\partial G_o}{\partial t'}\right|_{t'=t_o} \\
& + \frac{1}{4\pi}\int_{t_o}^{t_1}\mathrm{d}t'\oint_S \mathrm{d}a'\,[(\hat{\boldsymbol{n}}\times(\nabla'\times\boldsymbol{B}'))_S G_o + (\hat{\boldsymbol{n}}\times\boldsymbol{B}')_S\times\nabla'G_o + (\hat{\boldsymbol{n}}\cdot\boldsymbol{B}')_S\nabla'G_o]
\end{aligned} \tag{78}$$

† The current density which generates the initial conditions at $t = t'$ is of the form

$$\boldsymbol{J}(\boldsymbol{x}',t') = \frac{1}{c^2}\left[(\boldsymbol{A}'(\boldsymbol{x}'))_T\delta(t') + \left(\frac{\partial\boldsymbol{A}'(\boldsymbol{x}')}{\partial t'}\right)_T\delta(t')\right].$$

This is the same as that which generates the initial conditions in the Lorentz gauge (Morse and Feshbach 1953).

where

$$(\boldsymbol{B}')_{T,S} = (\nabla' \times \boldsymbol{A}')_{T,S}.$$

Equation (78) is consistent with the field equation for $\boldsymbol{B}_\gamma$, (3*b*). From (7*a*) we have

$$\begin{aligned}
\boldsymbol{E}_\gamma(\boldsymbol{x},t) = &-\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3x'\, G_o\left(\nabla'\rho' + \frac{1}{c^2}\frac{\partial \boldsymbol{J}'}{\partial t'}\right) \\
&+ \frac{1}{4\pi c^2}\int_V \mathrm{d}^3x' \left(G_o\left(\frac{\partial \boldsymbol{E}'}{\partial t'}\right)_T - (\boldsymbol{E}')_T\frac{\partial G_o}{\partial t'}\right|_{t'=t_o} \\
&+ \frac{1}{4\pi}\int_{t_o}^{t_1} \mathrm{d}t' \oint_S \mathrm{d}a'\, [(\hat{\boldsymbol{n}}\times(\nabla'\times\boldsymbol{E}'))_S G_o + (\hat{\boldsymbol{n}}\times\boldsymbol{E}')_S\times\nabla' G_o + (\hat{\boldsymbol{n}}\cdot\boldsymbol{E}')_S\nabla' G_o] \\
&+ \frac{\gamma}{\alpha^2c^2}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3x'\, G\left(\nabla'\cdot\boldsymbol{J}' + \frac{\partial\rho'}{\partial t'}\right) \\
&+ \frac{\gamma}{4\pi\alpha^2c^3}\int_V \mathrm{d}^3x' \left(G\left(\frac{\partial \boldsymbol{\Phi}'}{\partial t'}\right)_T - (\boldsymbol{\Phi}')_T\frac{\partial G}{\partial t'}\right|_{t'=t_o} \\
&+ \frac{\gamma}{4\pi c}\int_{t_o}^{t_1} \mathrm{d}t' \oint_S \mathrm{d}a'\, [(\hat{\boldsymbol{n}}\times\boldsymbol{\Phi}')_S\times\nabla' G + (\hat{\boldsymbol{n}}\cdot\boldsymbol{\Phi}')_S\nabla' G]
\end{aligned} \tag{79}$$

where

$$\boldsymbol{\Phi}' = \frac{\partial}{\partial t'}\nabla' g_\gamma = \frac{\partial}{\partial t'}\nabla'\left(\nabla'\cdot\boldsymbol{A}' + \frac{1}{\alpha^2 c}\frac{\partial\phi'}{\partial t'}\right) \tag{80}$$

$$(\boldsymbol{E}')_{T,S} = \left(-\nabla\phi' - \frac{1}{c}\frac{\partial \boldsymbol{A}'}{\partial t'}\right)_{T,S}.$$

The first three terms on the right-hand side of (79) are consistent with the field equation for $\boldsymbol{E}$, (3*a*). Since $\boldsymbol{\Phi}' = \boldsymbol{0}$ because of the gauge condition, and the continuity equation holds, we retain an α-independent solution for $\boldsymbol{E}_\gamma$. We note that the last three terms on the right-hand side of (79) are consistent with

$$\nabla\cdot[\Box_c^2 \boldsymbol{A}_\gamma(\boldsymbol{x},t) + \gamma\nabla(\nabla\cdot\boldsymbol{A}_\gamma(\boldsymbol{x},t))] + \frac{1}{\alpha^2 c}\frac{\partial}{\partial t}(\Box_{\alpha c}^2\phi_\gamma) \equiv -\frac{4\pi}{\alpha^2 c}\left(\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t}\right) = 0.$$

Thus it appears that for any given set of initial or boundary conditions the classical electric and magnetic fields remain independent of α, which therefore remains an arbitrary parameter. We now note that the relationships which existed between $\boldsymbol{A}_\gamma$, ϕ_γ, and $\boldsymbol{A}_o$, ϕ_o, (61), no longer hold. These quantities are now related thus:

$$\begin{aligned}
\boldsymbol{A}_\gamma(\boldsymbol{x},t) &= \boldsymbol{A}_o(\boldsymbol{x},t) + \nabla[\chi(\boldsymbol{x},t) + \tilde{\chi}_S(\boldsymbol{x},t)] \\
\phi_\gamma(\boldsymbol{x},t) &= \phi_o(\boldsymbol{x},t) - \frac{1}{c}\frac{\partial}{\partial t}[\chi(\boldsymbol{x},t) + \tilde{\chi}_T(\boldsymbol{x},t)]
\end{aligned} \tag{81}$$

where

$$\tilde{\chi}_S = -\frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \oint_S \mathrm{d}a'\, \Omega(\boldsymbol{x},t\,||\,\boldsymbol{x}',t')[\boldsymbol{J}(\boldsymbol{x}',t')\cdot\hat{\boldsymbol{n}}] \tag{82}$$

$$\tilde{\chi}_T = -\frac{1}{c}\int_V \mathrm{d}^3x'\, (\Omega(\boldsymbol{x},t\,||\,\boldsymbol{x}',t')\rho(\boldsymbol{x}',t')\,|_{t'=t_o} \tag{83}$$

and $\chi(\boldsymbol{x}, t)$ is that of (61), defined in (62). Since $\tilde{\chi}_S \neq \tilde{\chi}_T$ the usual form of the gauge transformation is lost. For an initial-value boundary-surface problem however, (81) is exactly the transformation required for the open part of the complete solutions of the potentials (i.e. the integral over $t' \in [t_o, t_1]$ and the volume V enclosed by the surface S) to maintain α-independent magnetic and electric fields, the $\tilde{\chi}_T$ taking into account the initial disturbance of the potential fields and the $\tilde{\chi}_S$ the effect of the boundary surface.

6. α-Lorentz gauge invariance and special relativity

In this section we briefly examine the constraints that special relativity may put on α, i.e. we examine the equations of motion in covariant form. We shall use the complex Minkowski space, with the spacetime 4-vector $x_\mu = (\boldsymbol{x}, ict)$.

The Lorentz invariance of Maxwell's equations (1), (2), has been known from the time of Lorentz and Poincaré. The covariance of these equations may be deduced from the experimentally verified law of the invariance of electric charge. Thus (1), (2) and (3) are respectively, in covariant form,

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 4\pi J_\mu \tag{84}$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \tag{85}$$

$$\frac{\partial^2}{\partial x_\nu^2} F_{\lambda\mu} = -\frac{4\pi}{c}\left(\frac{\partial J_\mu}{\partial x_\lambda} - \frac{\partial J_\lambda}{\partial x_\mu}\right) \tag{86}$$

where $F_{\mu\nu}$ is the antisymmetric field tensor in Minkowski space. Equation (85) implies the existence of potentials A_μ

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}. \tag{87}$$

These A_μ are not unique;the theory is invariant under the gauge transformation

$$A'_\mu \rightarrow A_\mu + \frac{\partial \Lambda}{\partial x_\mu} \tag{88}$$

where $\Lambda \equiv \Lambda(\boldsymbol{x}, t)$ is a well behaved scalar (see §2). One then chooses a gauge condition that is Lorentz invariant, with A_μ a 4-vector. Clearly the α-Lorentz gauge condition destroys manifest covariance. We wonder however if this makes any difference, since in quantum mechanical terms the gauge condition is a description of the vacuum state through a Gupta–Bleuler-type constraint. We will return to this point later. One point of interest here is to consider the physics of the condition $\alpha = -1$. In this case the gauge condition (20) reverts to the Lorentz condition, and the field equations (27), to those of the Lorentz gauge (12). The Lagrangian densities (14) and (25) are now identical. However $\tilde{\boldsymbol{A}}_\alpha$ and $\tilde{\phi}_\alpha$ are not zero being a mixture of retarded and advanced waves propagating at velocity c. For $t \gg t'$ these quantities are time antisymmetric combinations of retarded and advanced waves. This is in contrast to the time symmetric combinations of the same in the Wheeler–Feynman formalism (Wheeler and Feynman

1945), that Cramer(1986) has recently used to re-interpret quantum mechanics in terms of 'transactions' between quantum states. We also note from (66), that $\tilde{\boldsymbol{J}}_\alpha$ and $\tilde{\phi}_\alpha$ are zero. Thus it seems that one possible way of looking at retarded/advanced wave combinations is in terms of an α-Lorentz-type photon/tachyon pair! We return to this point again below.

7. Conclusion

The Lagrangian density which describes the dynamics of the electromagnetic field interacting with an external conserved charge and current density, in the complete α-Lorentz gauge, is given by (25). This Lagrangian density is arbitrary with respect to two parameters, the variable lagrange multiplier $\lambda \equiv \lambda(\boldsymbol{x}, t)$, and the constant α contained in the α-Lorentz gauge condition. The value of λ has no effect either on the electromagnetic potentials, $\boldsymbol{A} = \boldsymbol{A}_\gamma$ and $\phi = \phi_\gamma$, or on the observable electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively. Even under a gauge transformation of the second kind, which must not affect the dynamics of the fields, the value of λ remains completely arbitrary, although we must restrict the nature of the gauge parameter (the Λ of (16)), to (28). The same degree of arbitrariness is not evidenced by α. The value of α profoundly alters the nature of the electromagnetic potentials $\boldsymbol{A}_\gamma$ and ϕ_γ from the more common case $\alpha = 1$, i.e. the Lorentz gauge. The scalar potential ϕ_γ propagates from the real external conserved charge and current density at velocity αc. It is localised in space, having a single wavefront. The nature of propagation of the vector potential $\boldsymbol{A}_\gamma$ is more complicated, especially around times $t \simeq t'$ when the longitudinal and transverse motions are coupled. This potential is not localised in space, there being two wavefronts, one propagating at speed c the other at speed αc. For $t \gg t'$ the motion is like that of an expanding 'smoke ring'. However, even though the value of α alters the nature of the electromagnetic potentials in this way from that of the Lorentz gauge, these differences are not transmitted to the observable electric and magnetic fields, their propagation being fully causal and at speed c. This is because the α-Lorentz potentials are related to the Lorentz potentials by a gauge transformation. The 'tachyon potentials' are contained completely within the parameters of this gauge transformation, the χ of (61), and the χ, $\tilde{\chi}_T$, $\tilde{\chi}_S$ of (81). Hence classical considerations alone put no restrictions on the value of α.

In conclusion, several points are worthy of note. It is interesting that in the calculation of the fourth-order Green function, the integral (37) most naturally separates in prolate spheroidal coordinates. These coordinates are ones which suit a three-body colliding system; indeed (37) is not unlike a quantum mechanical transition matrix element (see for example McCarroll (1961)). Like our comments in §6, this also suggests a possible interpretation of the physics involved here in terms of an α-Lorentz tachyon/photon collision. Such a suggestion could only be verified by formulating a non-relativistic quantum field theory of the classical system studied above. Such an investigation would be worthwhile for several other reasons. First, as is well known, the electromagnetic potentials play a much more important role in the quantum mechanical treatment of the interaction of the electromagnetic field with matter than in the classical case. Also it appears from (50) that an α-dependent term is associated with the transverse modes of vibration albeit only for times around $t \simeq t'$, these transverse modes of vibration being the 'observable' ones in the quantum mechanical sense of that word. If this were true we would then be forced to conclude that either a superluminal α-Lorentz tachyon exists or, if the invariant velocity of light is sacrosanct, that $\alpha = 1$. This latter case would indeed be remarkable, for we would have proved, from quantum mechanical considerations alone, that such an invariant velocity exists. How appropriate it would be if the oldest and simplest field theory of all provided a link between relativity and quantum mechanics. Such a possibility may not be as far-fetched as it seems if we remember that not only are Maxwell's equations intrinsically Lorentz invariant but also intrinsically quantum mechanical. The former case above, if correct, is also interesting in that it might provide the answer to why no-one has ever observed an α-Lorentz tachyon. The coupling together of the longitudinal and transverse modes of vibration is a transient effect, such that after a long period of time these modes uncouple, the α-dependent terms then only being associated with the unobservable longitudinal motion. Perhaps if at one time such a particle existed it has long since departed to the realm of the unobservable. Given that no gauge transformation exists which transforms between the Lorentz and α-Lorentz gauges, this departure would be irrevocable. In any case these questions, and that of the importance of the non-Lorentz invariance of the α-Lorentz gauge condition, can only be answered by doing a second quantisation on the Lagrangian density of the system (25).

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Appendix 1

This appendix contains properties of the various generalised functions mentioned above.

The delta function $\delta(x-y)$ is defined thus:

$$\delta(x-y) = \begin{cases} 0 & x \neq y \\ \to \infty & x \to y. \end{cases} \tag{A1.1}$$

The Heaviside step function $H(x-y)$ is defined thus:

$$H(x-y) = \begin{cases} 0 & x < y \\ \frac{1}{2} & x = y \\ 1 & x > y. \end{cases} \tag{A1.2}$$

The minimum function $M(x,y)$ is defined thus:

$$M(x,y) = \begin{cases} x & x < y \\ \frac{1}{2}(x+y) & x = y \\ y & x > y. \end{cases} \tag{A1.3}$$

$\delta(x-y)$, $H(x-y)$, $M(x,y)$ are related thus:

$$M(x,y) = yH(x-y) + xH(y-x) \tag{A1.4}$$

$$\frac{\partial}{\partial x} M(x,y) = H(y-x) \tag{A1.5}$$

$$\frac{\partial}{\partial x} H(x-y) = \delta(x-y) \tag{A1.6}$$

and have the following integral representations:

$$\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{A1.7}$$

$$= (xy)^{1/2}\int_0^{\infty} \mathrm{d}k\, k J_{1/2}(kx)J_{1/2}(ky) \tag{A1.8}$$

$$H(x-y) = (xy)^{1/2}\int_0^{\infty} \mathrm{d}k\, J_{1/2}(kx)J_{-1/2}(ky) \tag{A1.9}$$

$$M(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k^2}\, \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{A1.10}$$

$$= (xy)^{1/2}\int_0^{\infty} \frac{\mathrm{d}k}{k} J_{1/2}(kx)J_{1/2}(ky) \tag{A1.11}$$

where $J_m(kx)$ is a Bessel function of order m.

Appendix 2

This appendix contains transformation formulae for converting volume integrals to surface integrals. They are all equivalent forms of Green's theorem (A2.1), and may be derived from the divergence theorem.

For the scalars u, v and the vectors $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\int_V \mathrm{d}\tau\, [u\nabla^2 v - v\nabla^2 u] = \oint_S \mathrm{d}a\, [u(\hat{\boldsymbol{n}}\cdot\nabla v) - v(\hat{\boldsymbol{n}}\cdot\nabla u)] \tag{A2.1}$$

$$\int_V \mathrm{d}\tau\, [u\nabla^2 \boldsymbol{P} - \boldsymbol{P}\nabla^2 u] = \oint_S \mathrm{d}a\, [u(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{P} - \boldsymbol{P}(\hat{\boldsymbol{n}}\cdot\nabla u)] \tag{A2.2}$$

$$= \oint_S \mathrm{d}a\, [(\nabla\cdot(\boldsymbol{P}u))\hat{\boldsymbol{n}} + (\nabla\times(\boldsymbol{P}u))\times\hat{\boldsymbol{n}} - 2\boldsymbol{P}(\nabla u\cdot\hat{\boldsymbol{n}})] \tag{A2.3}$$

$$\int_V \mathrm{d}\tau\, [u\nabla(\nabla\cdot\boldsymbol{P}) - (\boldsymbol{P}\cdot\nabla)\nabla u] = \oint_S \mathrm{d}a\, [u(\nabla\cdot\boldsymbol{P})\hat{\boldsymbol{n}} + (\nabla u\times\boldsymbol{P})\times\hat{\boldsymbol{n}} - \boldsymbol{P}(\nabla u\cdot\hat{\boldsymbol{n}})] \tag{A2.4}$$

$$\int_V \mathrm{d}\tau\, [\boldsymbol{P}\times\nabla^2\boldsymbol{Q} + \boldsymbol{Q}\times\nabla^2\boldsymbol{P} = \oint_S \mathrm{d}a\, [\boldsymbol{P}\times(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{Q} + \boldsymbol{Q}\times(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{P}] \tag{A2.5}$$

$$\int_V \mathrm{d}\tau\, [\boldsymbol{P}\cdot\nabla(\nabla\cdot\boldsymbol{Q}) - \boldsymbol{Q}\cdot\nabla(\nabla\cdot\boldsymbol{P})] = \oint_S \mathrm{d}a\, [(\nabla\cdot\boldsymbol{Q})\boldsymbol{P} - (\nabla\cdot\boldsymbol{P})\boldsymbol{Q}]\cdot\hat{\boldsymbol{n}} \tag{A2.6}$$

$$\int_V \mathrm{d}\tau\, [\boldsymbol{P}\cdot\nabla\times(\nabla\times\boldsymbol{Q}) - \boldsymbol{Q}\cdot\nabla\times(\nabla\times\boldsymbol{P})] = \oint_S \mathrm{d}a\, [(\hat{\boldsymbol{n}}\times\boldsymbol{P}).(\nabla\times\boldsymbol{Q}) + (\hat{\boldsymbol{n}}\times(\nabla\times\boldsymbol{P}))\cdot\boldsymbol{Q}]. \tag{A2.7}$$

We can use these last two vector identities to solve (27*a*) above for any given set of initial and boundary conditions. Consider

$$\int_{t_0}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}'\, [\Theta_\gamma(\boldsymbol{x},t\,||\,\boldsymbol{x}',t')\cdot\nabla(\nabla\cdot\boldsymbol{A}_\gamma(\boldsymbol{x},t)) - \boldsymbol{A}_\gamma(\boldsymbol{x},t)\cdot\nabla(\nabla\cdot\Theta_\gamma(\boldsymbol{x},t\,||\,\boldsymbol{x}',t'))]. \tag{A2.8}$$

Using the properties of $\Theta_\gamma(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t')$, (30), and (A2.6) and (A2.7) above, the complete solution is

$$\begin{aligned}A_\gamma(\boldsymbol{x}, t) = {} & \frac{1}{c}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \Theta_\gamma(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t') \cdot \boldsymbol{J}(\boldsymbol{x}', t') \\ & + \frac{1}{4\pi c^2}\int_V \mathrm{d}^3\boldsymbol{x}' \left[\Theta_\gamma \cdot \left(\frac{\partial \boldsymbol{A}'}{\partial t'}\right)_T - (\boldsymbol{A}')_T \cdot \frac{\partial \Theta_\gamma}{\partial t}\right|_{t'=t_o} \\ & + \frac{1}{4\pi}\int_{t_o}^{t_1} \mathrm{d}t' \oint_S \mathrm{d}a' [\alpha^2(\nabla' \cdot \boldsymbol{A}')_S(\Theta_\gamma \cdot \hat{\boldsymbol{n}}) - \alpha^2(\nabla' \cdot \Theta_\gamma)(\boldsymbol{A}' \cdot \hat{\boldsymbol{n}})_S \\ & - \Theta_\gamma \cdot (\hat{\boldsymbol{n}} \times (\nabla' \times \boldsymbol{A}'))_S - (\nabla' \times \Theta_\gamma) \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{A}')_S]\end{aligned} \tag{A2.9}$$

where we have used the notation of §5 for the initial and boundary conditions.

Appendix 3

We show in this appendix that, under the simple boundary conditions of §2, Yang's expression for A_γ and ours are the same. According to Yang

$$\left(\frac{\partial \tilde{A}_\alpha}{\partial t}\right)_{\text{Yang}} = c(\nabla\phi_c^L - \nabla\phi_{\alpha c}^{\alpha L})$$

where

$$\tilde{A}_\alpha = A_\gamma - A_o$$

i.e. (52*a*). Hence

$$\begin{aligned}\left(\frac{\partial \tilde{A}_\alpha}{\partial t}\right)_{\text{Yang}} & = c\nabla \int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \left(\frac{\delta(\tau - R/c)}{R} - \frac{\delta(\tau - R/\alpha c)}{R}\right)\rho(\boldsymbol{x}', t') \\ & = -c\nabla \int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \, \tilde{G}_\alpha(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t')\, \rho(\boldsymbol{x}', t') \\ & = \frac{1}{c}\nabla\frac{\partial}{\partial t}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \, \Omega(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t')\, \nabla' \cdot \boldsymbol{J}(\boldsymbol{x}', t')\end{aligned}$$

where we have used the fact that $\rho(\boldsymbol{x}', t_o) = 0$, and the continuity equation (9)

$$= \frac{1}{c}\nabla\frac{\partial}{\partial t}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \, \nabla'\Omega(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t') \cdot \boldsymbol{J}(\boldsymbol{x}', t')$$

for an infinite boundary surface,

$$= \frac{1}{c}\frac{\partial}{\partial t}\int_{t_o}^{t_1} \mathrm{d}t' \int_V \mathrm{d}^3\boldsymbol{x}' \, \Theta_\gamma(\boldsymbol{x}, t \,||\, \boldsymbol{x}', t') \cdot \boldsymbol{J}(\boldsymbol{x}', t').$$

This proves the result.

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